Symbolic Computation of Laplace-Dirichlet Eigenvalues

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Motivation

In 2004, Grinfeld and Strang [4] posed the following boundary problem. *What is the series in 1/N for the simple Laplace eigenvalues λ_N on a regular polygon with N sides under the Dirichlet boundary condition?* The first four terms in the series were computed by hand in [5].

The function $u(r, theta)$ is defined using Bessel Functions J_i and the ith root of the zeroth Bessel Function ρ.

$$u(r,\theta)=\frac{J_0(\rho r)}{\sqrt{\pi}J_1(\rho)} \tag{1}$$

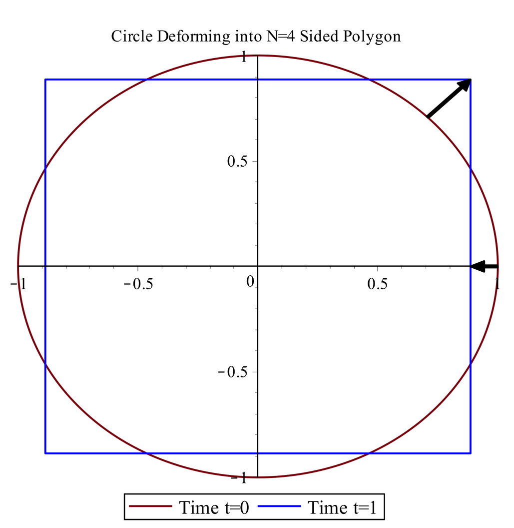
The Laplacian eigenvalue is

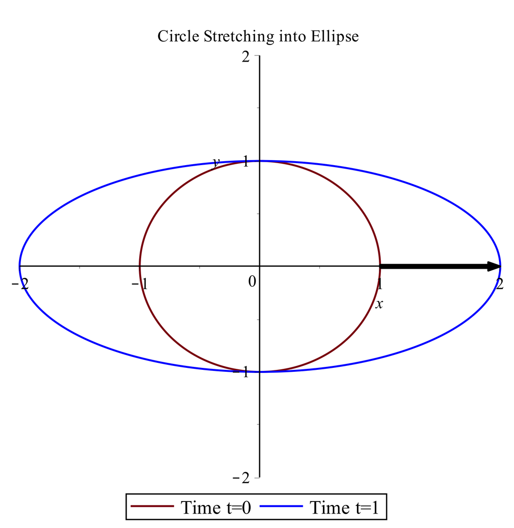
$$\nabla_i\nabla^i u=\lambda u \tag{2}$$

On the unit circle, under the Dirichlet boundary condition $u|_S=0$, $\lambda_{circle}=\rho^2$. To find λ_N, we deform the circle into a polygon with N sides and generate a Taylor Series. Solving for the fourth term required significant simplifications and the fifth term was intractable. This motivated the desire for a symbolic computation system that could handle this and similar problems.

Model Problems

To build a symbolic system, we selected model problems. A boundary variation of Poisson's equation was used to justify automation was possible.[1] We then examined a variation on the polygon problem, deforming the circle into an ellipse with semi-axes A = 1 and B = 1 + t?





Calculus of Moving Surfaces

The Calculus of Moving Surfaces (CMS) is an extension of Tensor Calculus to deforming manifolds. The CMS provides a mechanism for solving our problem by modeling the deformation of a circle into an ellipse. We can apply $\frac{\delta}{\delta t}$, the central operator of the CMS, to calculate each term in the series.

We automate the CMS using a Term Rewrite System (TRS) implementing equational properties as directional rules. The implementation of our TRS is described in [3]. After reduction by the TRS, expressions are evaluated by generating Maple code.

Some expressions are challenging to evaluate directly. The $\frac{\delta}{\delta t}$-derivative of mean curvature B^α_α can be replaced with an expression that is easier to evaluate $\nabla^\alpha\nabla_\alpha C+CB^\alpha_\beta B^\beta_\alpha$. The $\frac{\delta}{\delta t}$-derivative of $u(r,\theta)$ cannot be evaluated directly, replacing it with partial derivatives that we can calculate is crucial.

$$\frac{\delta u}{\delta t}\to\frac{\partial u}{\partial t}+CN_i\nabla^i u \tag{3}$$

Variations

Each term in $\lambda_{ellipse}$ has the format $\int L_n dS$. The first variation λ_1 is defined by Hadamard's formula.

$$\lambda_1=-\int_S C\nabla_i u\nabla^i u dS. \tag{4}$$

The CMS can be used to find the next term in the series.

$$\lambda_2=\frac{\delta}{\delta t}\left(-\int_S C\nabla_i u\nabla^i u dS\right) \tag{5}$$

The CMS provides a recursive formula for these expressions.

$$\lambda_n=\int L_n dS, L_n=\frac{\delta L_{n-1}}{\delta t}-CB^\alpha_\alpha L_{n-1} \tag{6}$$

Using this formula, we derive L_2

$$L_2=-\frac{\delta C\nabla_i u\nabla^i u}{\delta t}+CB^\alpha_\alpha C\nabla_i u\nabla^i u \tag{7}$$

L_2 is reduced to its normal form by our TRS. Rules are applied to ensure an expression where evaluation is possible. Some key rules are shown.

Applied Product Rule:

$$-\frac{\delta C\nabla^i u\nabla_i u}{\delta t}\to\quad -\frac{\delta C}{\delta t}\nabla^i u\nabla_i u-C\frac{\delta\nabla^i u}{\delta t}\nabla_i u-C\nabla^i u\frac{\delta\nabla_i u}{\delta t} \tag{8}$$

Applied Chain Rule:

$$\frac{\delta\nabla^i u}{\delta t}\to\quad\nabla^i\frac{\partial u}{\partial t}+CN^m\nabla_m\nabla^i u \tag{9}$$

The final normal form for L_2 is

$$\begin{aligned}L_2= \quad & C^2B^\alpha_\alpha\nabla^i u\nabla_i u-\frac{\delta C}{\delta\tau}\nabla^i u\nabla_i u-2C\nabla^i\frac{\partial u}{\partial t}\nabla_i u\\ & -2C^2N^m\nabla_i u\nabla^i\nabla_m u\end{aligned}$$

Using the recursive nature of the expression, further values can be calculated by the TRS.

$$\begin{aligned}L_3= \quad & -C^3B^\beta_\beta B^\alpha_\alpha\nabla^i u\nabla_i u+C\frac{\delta C}{\delta t}B^\alpha_\alpha\nabla^i u\nabla_i u\\
&+3C^2B^\alpha_\alpha\nabla^i\frac{\partial u}{\partial t}\nabla_i u+2C^3B^\alpha_\alpha N^j\nabla_i u\nabla^i\nabla_j u\\
&+2CB^\alpha_\alpha\frac{\delta C}{\delta t}\nabla^i u\nabla_i u+C^2\nabla^i u\nabla_i u\nabla^\alpha\nabla_\alpha C\\
&+C^3B^\alpha_\beta B^\beta_\alpha\nabla^i u\nabla_i u+2C^3B^\alpha_\alpha N^j\nabla_i u\nabla^i\nabla_j u\\
&+C^2B^\alpha_\alpha\nabla_i\frac{\partial u}{\partial t}\nabla^i u-\frac{\delta^2 C}{\delta^2 t}\nabla^i u\nabla_i u-\frac{\delta C}{\delta t}\nabla^i\frac{\partial u}{\partial t}\nabla_i u\\
&-2C\frac{\delta C}{\delta t}N^j\nabla_i u\nabla^i\nabla_j u-\frac{\delta C}{\delta t}\nabla_i\frac{\partial u}{\partial t}\nabla^i u-2\frac{\delta C}{\delta t}\nabla^i\frac{\partial u}{\partial t}\nabla_i u\\
&-2C\nabla^i\frac{\partial^2 u}{\partial^2 t}\nabla_i u-2C^2N^j\nabla_i u\nabla^i\nabla_j\frac{\partial u}{\partial t}-2C\nabla_i\frac{\partial u}{\partial t}\nabla^i\frac{\partial u}{\partial t}\\
&-4C^2N^j\nabla^i\frac{\partial u}{\partial t}\nabla_j\nabla_i u-4C\frac{\delta C}{\delta t}N^j\nabla_i u\nabla^i\nabla_j u\\
&+2C^2Z^j_\alpha\nabla_i u\nabla^i\nabla_j u\nabla^\alpha C-2C^3N^jN^k\nabla^i\nabla_j u\nabla_k\nabla_i u\\
&-2C^2N^j\nabla^i\nabla_j\frac{\partial u}{\partial t}\nabla_i u-2C^3N^jN^k\nabla_i u\nabla^i\nabla_k\nabla_j u\end{aligned} \tag{10}$$

The number of products in the summation after combining like terms are $L_1=1$, $L_2=4$, $L_3=23$, $L_4=137$, and $L_5=1154$. This rapid increase in the number of terms is a consistent feature of high order perturbation problems. It causes calculations to quickly becomes intractable.

To determine the series terms, these expressions are exported to Maple and evaluated on the surface.

Solutions

We present the first five variations. They are provided in terms of λ on the circle. We also provide the numeric approximation for each variation. The solutions have been confirmed to match numerical estimates.

$$\begin{aligned}
\lambda &= & \lambda &= 5.783\\
\lambda_1 &= & -\lambda &= -5.783\\
\lambda_2 &= & \tfrac{3}{2}\lambda+\tfrac{1}{4}\lambda^2 &= 17.036\\
\lambda_3 &= & -3\lambda-\tfrac{3}{2}\lambda^2 &= -67.517\\
\lambda_4 &= & \tfrac{15}{2}\lambda+\tfrac{15}{2}\lambda^2+\tfrac{87}{128}\lambda^3-\tfrac{21}{256}\lambda^4 &= 333.919\\
\lambda_5 &= & -\tfrac{45}{2}\lambda-\tfrac{75}{2}\lambda^2-\tfrac{1305}{128}\lambda^3+\tfrac{315}{256}\lambda^4 &= -1979.913
\end{aligned}$$

We give the first five terms in the series for the ellipse, in eccentricity ϵ

$$\begin{aligned}\lambda(\varepsilon)= \quad & \lambda+-\tfrac{1}{2}\lambda\varepsilon^2+\left(-\tfrac{3}{16}\lambda+\tfrac{1}{32}\lambda^2\right)\varepsilon^4+\left(-\tfrac{3}{32}\lambda+\tfrac{1}{64}\lambda^2\right)\varepsilon^6\\
&+\left(-\tfrac{7}{128}\lambda+\tfrac{3}{512}\lambda^2+\tfrac{29}{16384}\lambda^3-\tfrac{7}{32768}\lambda^4\right)\varepsilon^8\\
&+\left(-\tfrac{9}{256}\lambda+\tfrac{1}{1024}\lambda^2+\tfrac{87}{32768}\lambda^3-\tfrac{21}{65536}\lambda^4\right)\varepsilon^{10}\end{aligned} \tag{11}$$

The first seven terms and details of calculation will be presented in [2].

In the CMS, an expression is true for all coordinate systems. This means that our expressions are also valid for the N-sided polygon problem. The values of C and $\frac{\partial u}{\partial t}$ are infinite Fourier series in the N-sided polygon problem.

Conclusions

We have automated the computation of CMS problems that had previously been solved by hand. We have extended the results on these problems past the point at which they had previously become intractable. We have shown that the CMS has potential for automation using a TRS. We have proven that our system produces accurate results by using it on problems that can also be solved by numerical approximation.

We will now apply our system to the N-sided polygon problem, which motivated its development. Due to the nature of the CMS, our derived expressions are true for any coordinate system. We must now solve the problem of evaluating these expressions on the N-sided polygon. The remaining hurdle of this problem is the complexity of evaluation on the N-sided polygon.

References

[1] M. Boady, P. Grinfeld, and J. Johnson. Boundary variation of Poisson's equation: a model problem for symbolic calculus of moving surfaces. *Int. J. Math. Comp. Sci.*, 6(2), 2011.

[2] M. Boady, P. Grinfeld, and J. Johnson. Laplace eigenvalues on the ellipse and the symbolic calculus of moving surfaces. *In preparation.*

[3] M. Boady, P. Grinfeld, and J. Johnson. A Term Rewriting System for the Calculus of Moving Surfaces. *International Symposium on Symbolic and Algebraic Computation (ISSAC)*, 2013.

[4] P. Grinfeld and G. Strang. Laplace eigenvalues on polygons. *Computers and Mathematics with Applications* 48:1121–1133, 2004.

[5] P. Grinfeld and G. Strang. Laplace eigenvalues on regular polygons: A series in 1/N. *Journal of Mathematical Analysis and Applications* 385(1):135–149, 2012.