

MOTIVATION

In 2004, Grinfeld and Strang [3] posed the following boundary problem. *What is the series in $1/N$ for the simple Laplace eigenvalues λ_N on a regular polygon with N sides under the Dirichlet boundary condition?* The first four terms in the series were computed by hand in 2012 [4]. Our goal is to automate the process and derive more terms in the series.

The Calculus of Moving Surfaces (CMS), an extension of tensor calculus to deforming manifold, provides an analytic method to describe the change caused by the deformation of a surface. The Laplace eigenvalues on the unit circle under the Dirichlet boundary condition are known. We have applied the CMS to analyze the change in the eigenvalues between the unit circle and N -sided regular polygon[2]. It remains to evaluate these expressions and form a Taylor series for the eigenvalues on the N -sided polygon.

This problem is interesting in applied mathematics[5], physics[1] and pattern recognition[6]. It also presents a number of analytical challenges associated with the singularities present in the problem. This problem has important implications for the analysis of numerical methods in which a smooth boundary is replaced by an approximated shape.

SURFACE DEFORMATION

The Laplace-Dirichlet Eigenvalue problem is described by the following set of equations for a boundary S embedded in space Ω .

$$\Delta u = -\lambda u \quad (1)$$

$$u|_S = 0 \quad (2)$$

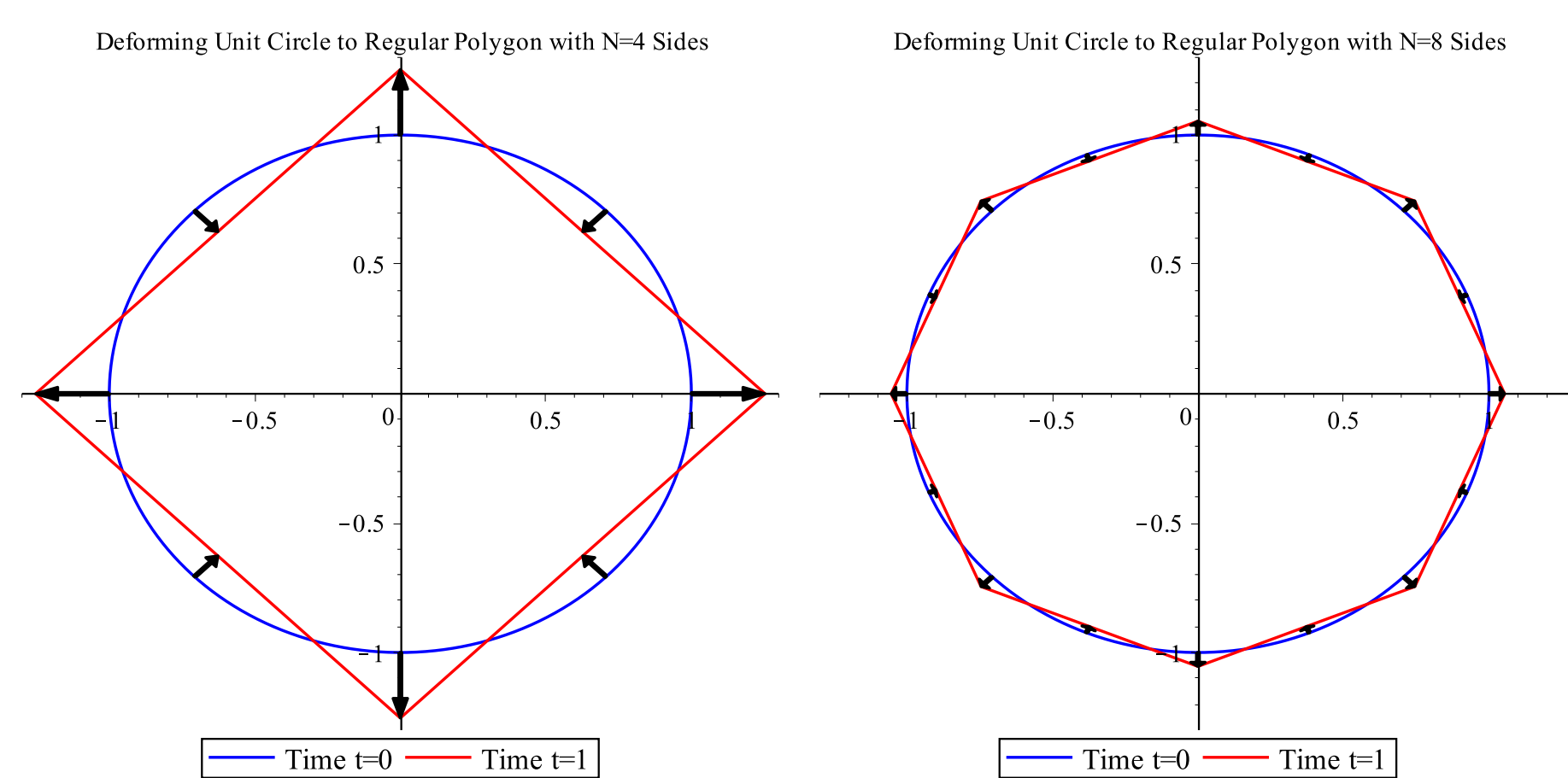
$$\int_{\Omega} u^2 d\Omega = 1 \quad (3)$$

The unknowns, u and λ , can be found when S is the unit circle and Ω is described in polar coordinates.

$$u(r, \theta) = \frac{J_0(\rho r)}{\sqrt{\pi} J_1(\rho)} \quad (4)$$

$$\lambda = \rho^2 \quad (5)$$

In this expression, J_m is the m -th Bessel Function and ρ is the n -th root of J_0 . The surface will start at $t = 0$ with the unit circle and deform into an N -sided regular polygon at time $t = 1$. The area will be constant for the deformation.



We will use a Taylor series to describe $\lambda(t = 1)$ in terms of the i -th derivatives of $\lambda_i(t = 0) = \lambda_i$.

$$\lambda(t = 1) = \lambda + \lambda_1 + \frac{1}{2!} \lambda_2 + \frac{1}{3!} \lambda_3 + \frac{1}{4!} \lambda_4 + \dots \quad (6)$$

EVALUATION

The CMS provides expressions for the derivatives of λ but evaluation is dependent on the surface.

$$\lambda_1 = - \int_S C \nabla_i u \nabla^i u dS. \quad (7)$$

$$\lambda_2 = \int_S (C^2 B_\alpha^\alpha \nabla^i u \nabla_i u - \frac{\delta C}{\delta \tau} \nabla^i u \nabla_i u - 2C \nabla^i \frac{\partial u}{\partial t} \nabla_i u - 2C^2 N^m \nabla_i u \nabla^i \nabla_m u) dS \quad (8)$$

The surface deformation is encapsulated in the surface velocity, C , and the partial derivatives of u , $\frac{\partial u}{\partial t}$. The terms only need to be evaluated at $t = 0$.

$$C|_{t=0} = \frac{-\cos(\theta) + \cos(\pi/N)}{\cos(\theta)} \quad (9)$$

To work with this expression we represent it as a Fourier series, $\sum_{k=-\infty}^{\infty} c_0(k) e^{ikN\theta}$ where

$$c_0(k) = \begin{cases} -\frac{\pi^2}{3N^2} & k = 0 \\ \frac{(-1)^k \pi^2}{N^2 k^2} + \frac{(-1)^k \pi^2}{3N^4 k^4} - \frac{5(-1)^k}{N^4 k^4} + \dots & otherwise \end{cases} \quad (10)$$

Next, we evaluate λ_1 in terms of the Fourier coefficients.

$$\lambda_1 = -2c_0(0)\rho^2 = \frac{2\pi^2 \rho^2}{3N^2} \quad (11)$$

For λ_2 , the result includes convolutions of coefficients.

$$\lambda_2 = 2conv(c_0, c_0)(0)\rho^2 - 2c_1(0)\rho^2 + 4\sqrt{\pi} conv\left(c_0, \frac{\delta u_1}{\delta r}\bigg|_{r=1}\right)(0) \quad (12)$$

Where $u_1(k)$, is the k -th coefficient of the Fourier series for $\frac{\partial u}{\partial t}$. The expressions for u_i are derived from the CMS and will also include convolutions of the c_i coefficients.

$$u_1(k) = \begin{cases} \frac{c_0(0)(\rho r J_1(\rho r) - J_0(\rho r))}{\sqrt{\pi} J_1(\rho)} & k = 0 \\ \frac{c_0(k) \rho J_{kN}(\rho r)}{J_{kN}(\rho) \sqrt{\pi}} & otherwise \end{cases} \quad (13)$$

By approximating these convolutions, we can determine the solution to λ_2 .

$$\lambda_2 = \frac{8\zeta(3)\rho^2}{N^3} + \frac{32\pi^4 \rho^2}{45N^4} + O(N^{-5}) \quad (14)$$

CONVOLUTIONS

Convolutions are approximated using a custom Maple library.

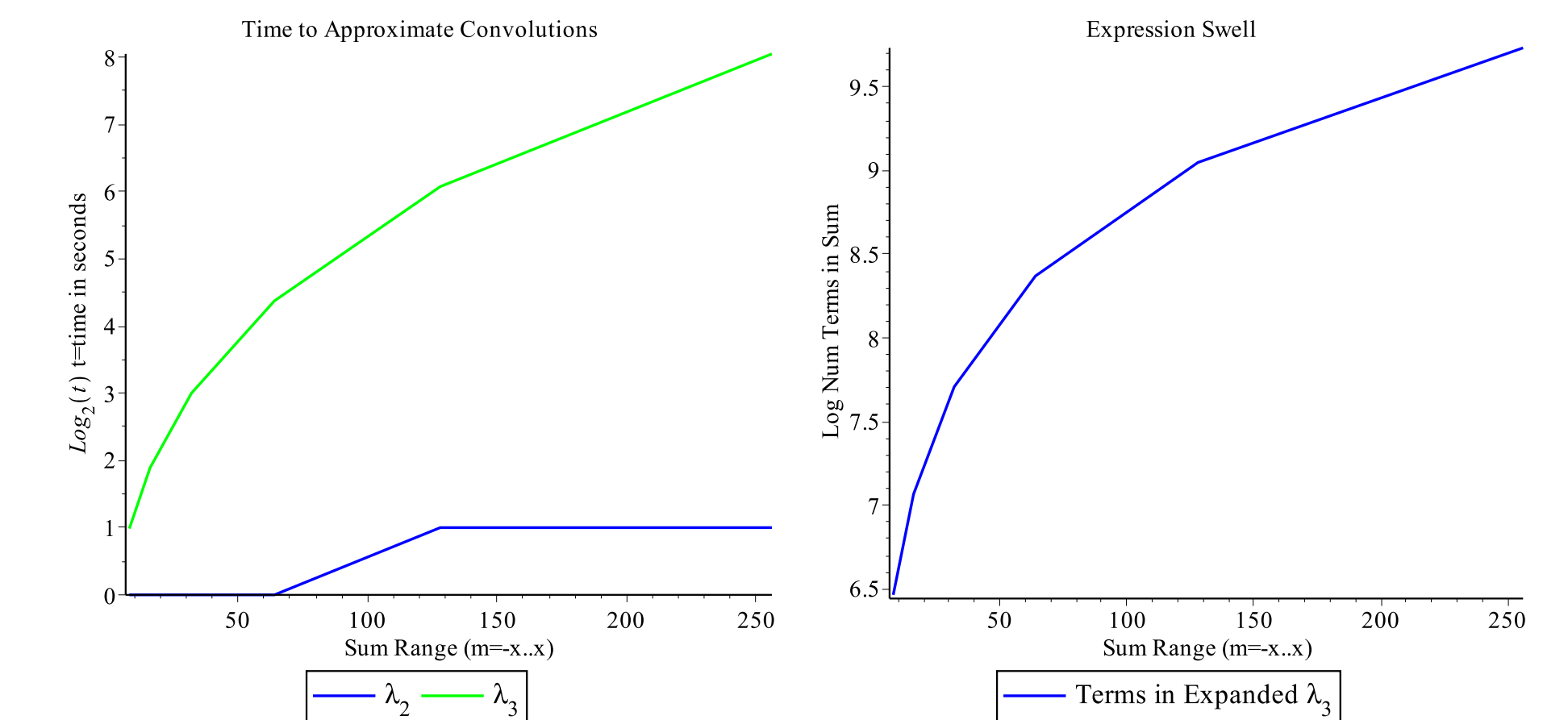
$$conv(a, b)(i) = \sum_{k=-m}^m a(k)b(i-k) \quad (15)$$

Deeper convolutions require more terms approximate.

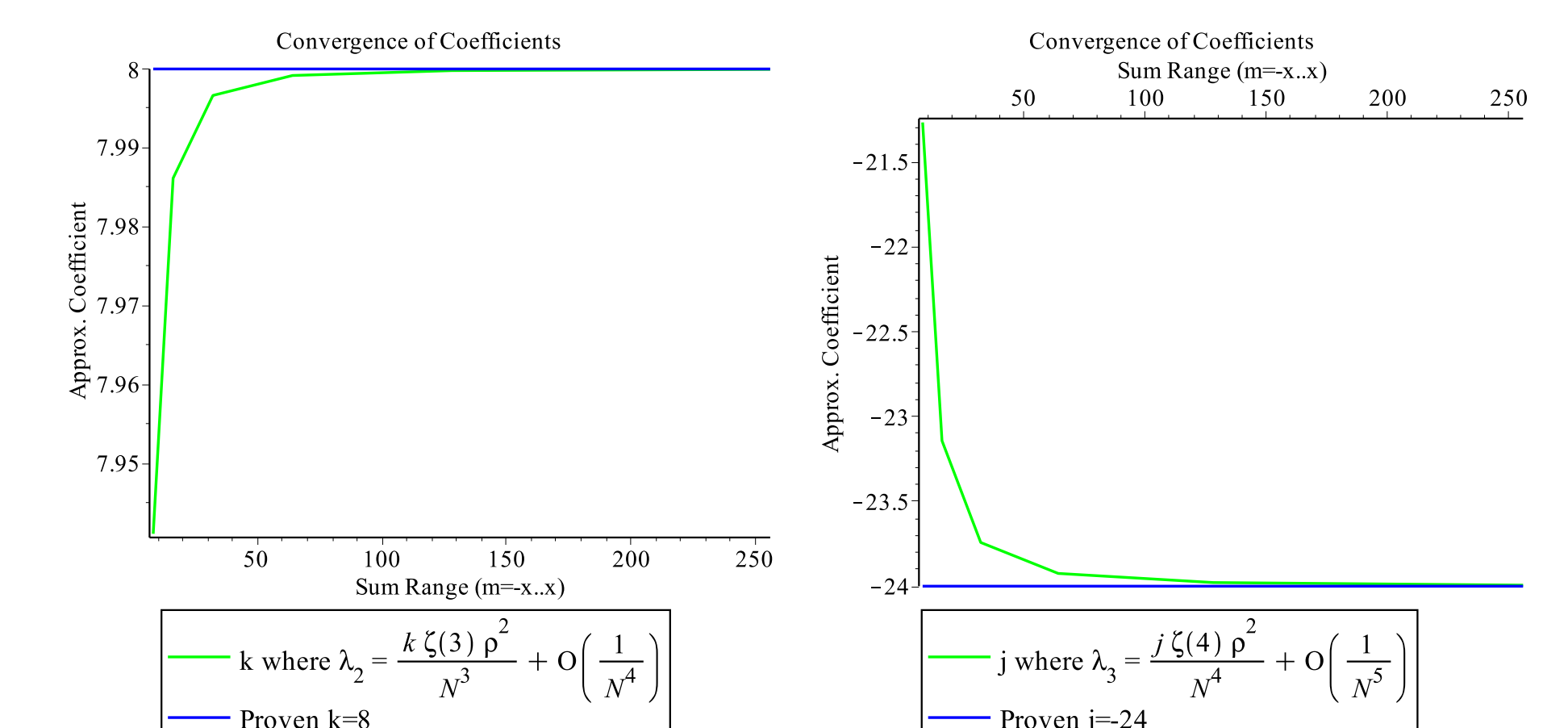
$$conv(c, conv(a, b))(i) = \sum_{j=-m}^m c(j) \left(\sum_{k=-m}^m a(k)b((i-j)-k) \right) \quad (16)$$

RESULTS

Evaluating more terms increases accuracy but also the time and number of terms that need to be simplified.



We believe that all values will converge to integer multiple of the ζ function in the form $\frac{i\zeta(j)\rho^k}{N^j}$ for integers i, j , and k .



Our current Taylor series up to $\frac{1}{N^5}$ is

$$\lambda(t = 1) = \rho^2 + 2/3 \frac{\pi^2 \rho^2}{N^2} + 4 \frac{\zeta(3) \rho^2}{N^3} + \frac{14 \pi^4 \rho^2}{45 N^4} + \frac{1}{22680} \frac{\rho^2 (-45360 \zeta(5) \rho^2 + 453600 \zeta(5))}{N^5} \quad (17)$$

CHALLENGES

Although the results are simple, the expression swells rapidly during evaluation. The highest we can currently determine λ_4 is with convolutions of range $m = -32 \dots 32$. For this sum, we need to evaluate 22,494 terms which took over an hour. Our goal is to optimize the calculations using methods such as parallel processing to solve λ_4 for range $m = -256 \dots 256$ in minutes.

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