Symbolic Computation of Laplace-Dirichlet Eigenvalues

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Overview

The Calculus of Moving Surfaces (CMS) is an analytic framework that extends the Tensor Calculus to deforming manifolds. This includes problems in boundary variation, fluid films, and shape optimization. No symbolic packages exist to manipulate expressions in the CMS. The analytic framework of the CMS is coordinate free. Expressions in the CMS can be evaluated in any defined coordinate system. We have developed two packages to solve problems in the CMS. A Term Rewrite System (TRS) focuses on the high level analytic framework and our Maple package handles direct calculations in specific coordinate systems.

Our initial focus is on boundary variation problems. We have used our two systems to determine Laplace-Dirichlet Eigenvalues. This is a problem of current interest to researchers studying the CMS.

Laplace-Dirichlet Eigenvalues

The Laplace-Dirichlet Eigenvalue problem is defined by the system of equations

$$\nabla_i \nabla^i (u) = -\lambda u, u|_S = 0 \tag{1}$$

∇_i is the covariant derivative and ∇^i is the contravariant derivative. The field u is defined over the entire problem space and $u|_S$ is the part of the field that exists on the surface manifold.

The deforming manifold in this problem is a ellipse with semi-axis $A = 1 + c$ and $B = 1$. The surface begins as a circle at time $t = 0$. The circle deforms into an ellipse.

The initial values for u and λ are given by

$$\begin{aligned} u(r,\theta) &= \frac{J_0(\rho r)}{\sqrt{\pi} J_1(\rho)} \\ \lambda &= \rho^2 \end{aligned} \tag{2}$$

J_m is the m-th Bessel function and ρ is the n-th root of J_0. The variables r and θ describe a point in 2 dimensional space using polar coordinates.

The goal of the problem is to find the variations of λ. We define λ_n to be the nth variation of λ. These terms provide a partial series for the eccentricity ϵ for the lowest eigenvalue $\lambda(\epsilon)$.

Term Rewrite System

A TRS transforms an expression based on a set of reduction rules. The goal of the TRS is to convert an given expression into a normal form. For this problem, the normal form can be directly calculated.

$$\frac{\delta F}{\delta t} \rightarrow \frac{\partial F}{\partial t} + C N_i \nabla^i F \tag{3}$$

This rule is true for any spacial field F. In this problem, the value $\frac{\delta u}{\delta t}$ cannot be calculated directly. Application of this rule to u in provides a calculable expression.

Variations

Each Lambda has the format $\int L_n dS$. The first variation λ_1 is defined by the problem.

$$\lambda_1 = -\int_S C \nabla_i u \nabla^i u dS. \tag{4}$$

All future λ expressions can be derived from the recursive formula.

$$L_n = \frac{\delta L_{n-1}}{\delta \tau} - C B^\alpha_\alpha L_{n-1} \tag{5}$$

$$\lambda_n = \int L_n dS \tag{6}$$

Using this formula, we derive L_2

$$L_2 = -\frac{\delta C \nabla_i u \nabla^i u}{\delta \tau} + C B^\alpha_\alpha C \nabla_i u \nabla^i u \tag{7}$$

L_2 is reduced to its normal form by our TRS. Some key rule applications are shown.

Applied Product Rule:

$$\begin{aligned} -\frac{\delta(1) C \nabla^i u \nabla_i u}{\delta t} \rightarrow \quad & -\frac{\delta 1}{\delta t} C \nabla^i u \nabla_i u - \frac{\delta C}{\delta t} \nabla^i u \nabla_i u \\ & -C \frac{\delta \nabla^i u}{\delta t} \nabla_i u - C \nabla^i u \frac{\delta \nabla_i u}{\delta t} \end{aligned} \tag{8}$$

Applied Chain Rule:

$$\frac{\delta \nabla^i u}{\delta t} \rightarrow \quad \nabla^i \frac{\partial u}{\partial t} + C N^m \nabla_m \nabla^i u \tag{9}$$

The final normal form for L_2 is

$$\begin{aligned} L_2 = \quad & C^2 B^\alpha_\alpha \nabla^i u \nabla_i u - \frac{\delta C}{\delta \tau} \nabla^i u \nabla_i u - 2C \nabla^i \frac{\partial u}{\partial t} \nabla_i u \\ & -2C^2 N^m \nabla_i u \nabla^i \nabla_m u \end{aligned} \tag{10}$$

Using the recursive nature of the expression, further values can be calculated by the TRS.

$$\begin{aligned} L_3 = \quad & -C^3 B^\beta_\beta B^\alpha_\alpha \nabla^i u \nabla_i u + C \frac{\delta C}{\delta t} B^\alpha_\alpha \nabla^i u \nabla_i u \\ & +3C^2 B^\alpha_\alpha \nabla^i \frac{\partial u}{\partial t} \nabla_i u + 2C^3 B^\alpha_\alpha N^j \nabla_i u \nabla^i_j u \\ & +2C B^\alpha_\alpha \frac{\delta C}{\delta t} \nabla^i u \nabla_i u + C^2 \nabla^i u \nabla_i u \nabla^\alpha_\alpha C \\ & +C^3 B^\alpha_\beta B^\beta_\alpha \nabla^i u \nabla_i u + 2C^3 B^\alpha_\alpha N^j \nabla_i u \nabla^i_j u \\ & +C^2 B^\alpha_\alpha \nabla_i \frac{\partial u}{\partial t} \nabla^i u - \frac{\delta^2 C}{\delta^2 t} \nabla^i u \nabla_i u - \frac{\delta C}{\delta t} \nabla^i \frac{\partial u}{\partial t} \nabla_i u \\ & -2C \frac{\delta C}{\delta t} N^j \nabla_i u \nabla^i_j u - \frac{\delta C}{\delta t} \nabla_i \frac{\partial u}{\partial t} \nabla^i u - 2 \frac{\delta C}{\delta t} \nabla^i \frac{\partial u}{\partial t} \nabla_i u \\ & -2C \nabla^i \frac{\partial^2 u}{\partial^2 t} \nabla_i u - 2C^2 N^j \nabla_i u \nabla^i_j \frac{\partial u}{\partial t} - 2C \nabla_i \frac{\partial u}{\partial t} \nabla^i \frac{\partial u}{\partial t} \\ & -4C^2 N^j \nabla^i \frac{\partial u}{\partial t} \nabla_{ji} u - 4C \frac{\delta C}{\delta t} N^j \nabla_i u \nabla^i_j u \\ & +2C^2 Z^j_\alpha \nabla_i u \nabla^i_j u \nabla^\alpha C - 2C^3 N^j N^k \nabla^i_j u \nabla_{ki} u \\ & -2C^2 N^j \nabla^i_j \frac{\partial u}{\partial t} \nabla_i u - 2C^3 N^j N^k \nabla_i u \nabla^i_{kj} u \end{aligned} \tag{11}$$

Number of products in the summation for each L_n is $L_1 = 1$, $L_2 = 4$, $L_3 = 23$, $L_4 = 137$, and $L_5 = 1154$.

Solutions

We have determined the first five variations exactly. They are provided in terms of the initial λ. The variations are evaluated at $t = 0$. We also provide the numeric approximation for each variation. The solutions have been confirmed to match numerical estimates for up to 29 digits of accuracy.

$$\begin{aligned} \lambda &= & \lambda & &= 5.783 \\ \lambda_1 &= & -\lambda & &= -5.783 \\ \lambda_2 &= & \tfrac{3}{2}\lambda + \tfrac{1}{4}\lambda^2 & &= 17.036 \\ \lambda_3 &= & -3\lambda - \tfrac{3}{2}\lambda^2 & &= -67.517 \\ \lambda_4 &= & \tfrac{15}{2}\lambda + \tfrac{15}{2}\lambda^2 + \tfrac{87}{128}\lambda^3 - \tfrac{21}{256}\lambda^4 & &= 333.919 \\ \lambda_5 &= & -\tfrac{45}{2}\lambda - \tfrac{75}{2}\lambda^2 - \tfrac{1305}{128}\lambda^3 + \tfrac{315}{256}\lambda^4 & &= -1979.913 \end{aligned}$$

Using these results we give the first five terms in the series in eccentricity e for the lowest eigenvalue $\lambda(\varepsilon)$

$$\begin{aligned} \lambda(\varepsilon) = \quad & \lambda + -\tfrac{1}{2}\lambda \varepsilon^2 + \left(-\tfrac{3}{16}\lambda + \tfrac{1}{32}\lambda^2\right)\varepsilon^4 + \left(-\tfrac{3}{32}\lambda + \tfrac{1}{64}\lambda^2\right)\varepsilon^6 \\ & + \left(-\tfrac{7}{128}\lambda + \tfrac{3}{512}\lambda^2 + \tfrac{29}{16384}\lambda^3 - \tfrac{7}{32768}\lambda^4\right)\varepsilon^8 \\ & + \left(-\tfrac{9}{256}\lambda + \tfrac{1}{1024}\lambda^2 + \tfrac{87}{32768}\lambda^3 - \tfrac{21}{65536}\lambda^4\right)\varepsilon^{10} \end{aligned} \tag{12}$$

Previously this series was only correctly established up to the second order. The seminal work [4] gives a third order expression but this term is incorrect.

Conclusions

Our work has accomplished two goals.

1. We have shown that simple exact variations to the Laplace-Dirichlet Eigenvalue problem exist and can be determined. This alone is significant for researchers in the CMS.
2. We have shown that the CMS can be automated. This automation can be used to solve problems beyond the scope of hand calculations.

References

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