# Symbolic Computation of Laplace-Dirichlet Eigenvalues

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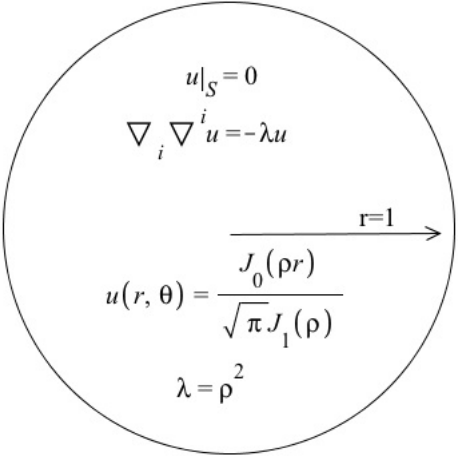
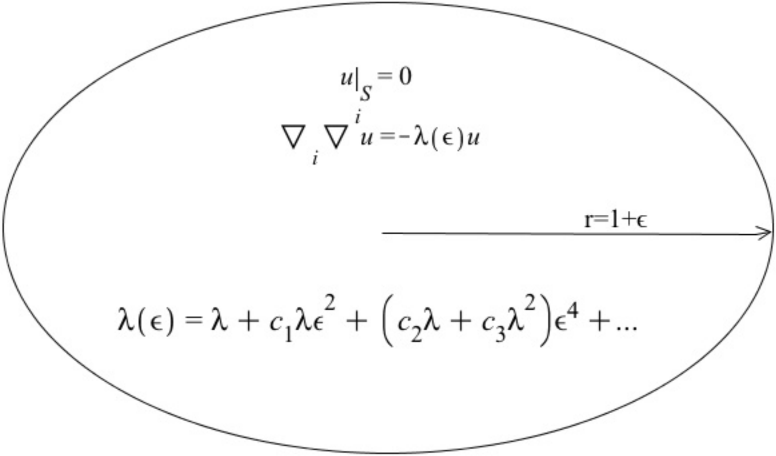
May 4, 2012

# Problem


$u|_S = 0$
$\nabla_i \nabla^i u = -\lambda u$
r=1
$u(r, \theta) = \frac{J_0(\rho r)}{\sqrt{\pi} J_1(\rho)}$
$\lambda = \rho^2$

# Problem


$u|_S = 0$
$\nabla_i \nabla^i u = -\lambda(\epsilon) u$
r=1+ϵ
$\lambda(\epsilon) = \lambda + c_1 \lambda \epsilon^2 + \left(c_2 \lambda + c_3 \lambda^2\right) \epsilon^4 + \ldots$

# Calculus of Moving Surfaces

- The Calculus of Moving Surfaces is an extenstion of Tensor Calculus
- A tensor is an n-dimensional array
- Tensors can exist on both the general space and specific surfaces
- Surfaces change shape over time
- Variations are derivatives with respect to the change of the surface over time
- The terms in our summation come from the varations of the original lambda

# Term Rewriting Systems

- Term Rewriting Systems automate the process of transforming an expression given rules.
- $\frac{\delta F}{\delta t} \rightarrow \frac{\partial F}{\partial t} + CN_i \nabla^i F$
- Used to simplify a CMS expression into a Calculatable form.
- Lambda's variation are defined by the CMS
- $\lambda_1 = -\int_S C\nabla_i u \nabla^i u dS$
- $\lambda_n = \int L_n dS$
- A recursive formula is used to determine each variation
- $L_n = \frac{\delta L_{n-1}}{\delta \tau} - CB^\alpha_\alpha L_{n-1}$
- $L_2 = -\frac{\delta C\nabla_i u \nabla^i u}{\delta \tau} + CB^\alpha_\alpha C\nabla_i u \nabla^i u$

# Rewriting the Expression

Applied Product Rule:

$$-\frac{\delta(1)C\nabla^i u\nabla_i u}{\delta t} \rightarrow \begin{array}{l} -\frac{\delta 1}{\delta t}C\nabla^i u\nabla_i u - \frac{\delta C}{\delta t}\nabla^i u\nabla_i u \\ -C\frac{\delta\nabla^i u}{\delta t}\nabla_i u - C\nabla^i u\frac{\delta\nabla_i u}{\delta t} \end{array}$$

Applied Chain Rule:

$$\frac{\delta\nabla^i u}{\delta t} \rightarrow \nabla^i\frac{\partial u}{\partial t} + CN^m\nabla_m\nabla^i u$$

The final normal form for $L_2$ is

$$L_2 = \begin{array}{l} C^2 B_\alpha^\alpha\nabla^i u\nabla_i u - \frac{\delta C}{\delta\tau}\nabla^i u\nabla_i u - 2C\nabla^i\frac{\partial u}{\partial t}\nabla_i u \\ -2C^2 N^m\nabla_i u\nabla^i\nabla_m u \end{array}$$

# Lambda 3

$$
\begin{aligned}
L_3 = \; & -C^3 B^\beta_\beta B^\alpha_\alpha \nabla^i u \nabla_i u + C\frac{\delta C}{\delta t} B^\alpha_\alpha \nabla^i u \nabla_i u \\
& +3C^2 B^\alpha_\alpha \nabla^i \frac{\partial u}{\partial t}\nabla_i u + 2C^3 B^\alpha_\alpha N^j \nabla_i u \nabla^i_j u \\
& +2CB^\alpha_\alpha \frac{\delta C}{\delta t}\nabla^i u\nabla_i u + C^2\nabla^i u \nabla_i u \nabla^\alpha_\alpha C \\
& +C^3 B^\alpha_\beta B^\beta_\alpha \nabla^i u\nabla_i u + 2C^3 B^\alpha_\alpha N^j \nabla_i u \nabla^i_j u \\
& +C^2 B^\alpha_\alpha \nabla_i \frac{\partial u}{\partial t}\nabla^i u - \frac{\delta^2 C}{\delta^2 t}\nabla^i u\nabla_i u - \frac{\delta C}{\delta t}\nabla^i\frac{\partial u}{\partial t}\nabla_i u \\
& -2C\frac{\delta C}{\delta t}N^j\nabla_i u\nabla^i_j u - \frac{\delta C}{\delta t}\nabla_i\frac{\partial u}{\partial t}\nabla^i u - 2\frac{\delta C}{\delta t}\nabla^i\frac{\partial u}{\partial t}\nabla_i u \\
& -2C\nabla^i\frac{\partial^2 u}{\partial^2 t}\nabla_i u - 2C^2 N^j\nabla_i u\nabla^i_j\frac{\partial u}{\partial t} - 2C\nabla_i\frac{\partial u}{\partial t}\nabla^i\frac{\partial u}{\partial t} \\
& -4C^2N^j\nabla^i\frac{\partial u}{\partial t}\nabla_{ji}u - 4C\frac{\delta C}{\delta t}N^j\nabla_i u\nabla^i_j u \\
& +2C^2 Z^j_\alpha \nabla_i u\nabla^i_j u\nabla^\alpha C - 2C^3 N^j N^k \nabla^i_j u \nabla_{ki} u \\
& -2C^2 N^j \nabla^i_j \frac{\partial u}{\partial t}\nabla_i u - 2C^3 N^j N^k \nabla_i u \nabla^i_{kj} u
\end{aligned}
$$

# Results

Solutions for the first 5 varations of the original $\lambda$

$$\lambda_1 = -\lambda$$
$$\lambda_2 = \frac{3}{2}\lambda + \frac{1}{4}\lambda^2$$
$$\lambda_3 = -3\lambda - \frac{3}{2}\lambda^2$$
$$\lambda_4 = \frac{15}{2}\lambda + \frac{15}{2}\lambda^2 + \frac{87}{128}\lambda^3 - \frac{21}{256}\lambda^4$$
$$\lambda_5 = -\frac{45}{2}\lambda - \frac{75}{2}\lambda^2 - \frac{1305}{128}\lambda^3 + \frac{315}{256}\lambda^4$$

These provide a partial series $\lambda(\epsilon)$

$$\lambda(\epsilon) = \lambda - \frac{1}{2}\lambda\epsilon^2 + \left(-\frac{3}{16}\lambda + \frac{1}{32}\lambda^2\right)\epsilon^4 + \left(-\frac{3}{32}\lambda + \frac{1}{64}\lambda^2\right)\epsilon^6 + \left(-\frac{7}{128}\lambda + \frac{3}{512}\lambda^2 + \frac{29}{16384}\lambda^3 - \frac{7}{32768}\lambda^4\right)\epsilon^8 + \left(-\frac{9}{256}\lambda + \frac{1}{1024}\lambda^2 + \frac{87}{32768}\lambda^2 - \frac{21}{65536}\lambda^4\right)\epsilon^{10}$$