A Symbolic Computation System for the Calculus of Moving Surfaces

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Objective

Many systems exist that allow symbolic expressions to be simplified or solved. Programs like Maple and Mathematica as well as some graphic calculators can handle expressions symbolically. These systems allow users to solve problems without tedious hand calculations. For example, users can quickly determine a symbolic derivative.

$$\frac{d}{dx}\left(x^2\cos(x)\right) = 2x\cos(x) - x^2\sin(x)$$

The Calculus of Moving Surfaces (CMS) is an extension of Tensor Calculus on stationary surfaces to moving surfaces. In these systems, forces exist on the surface, which is also moving or changing shape.

No programs are currently available to symbolically solve problems in the Calculus of Moving Surfaces. Many research fields take advantage of the CMS and would benefit from a symbolic system. Our research is in the development of a system to fill this gap and help advance ongoing research in the CMS.

Motivation

The CMS provides analytic tools for finding solutions to a wide range of problems with moving surfaces such as Fluid Film Dynamics, Boundary Variation Problems, and Shape Optimization Problems.

To determine the viability of the project, we selected a subset of problems that can be solved in a similar way. This limited the number of rules and objects that needed to be implemented, while still allow reasonable problems to be solved. These problems are related to boundary variations and similar to ongoing research in the CMS.

A model problem considers Poisson's equation on an N-sided polygon. The boundary variation condition is introduced by changing from a circle to a polygon with N sides. The CMS then allows for the determination of elements in the series for the Poisson energy E_N and formulas for variations of arbitrary order u_i. Solutions to this problem are know, allowing our results to be tested. The problem uses all the rules needed for similar problems that are of current interest. This will allow the system to be used on these problems when it can accurately solve the model problem.

A symbolic system will solve problems with existing methods and improve research into the CMS. As with any analytic framework, the complexity of calculations grows rapidly with the order of approximation. This means that hand calculations quickly become error prone or intractable. Automated symbolic computations will not make errors or become hindered by complex calculations. Symbolic Computation also offers advantages over numerical methods, particularly when the boundary perturbation is too complex to be captured effectively and when the perturbation leads to singularities.

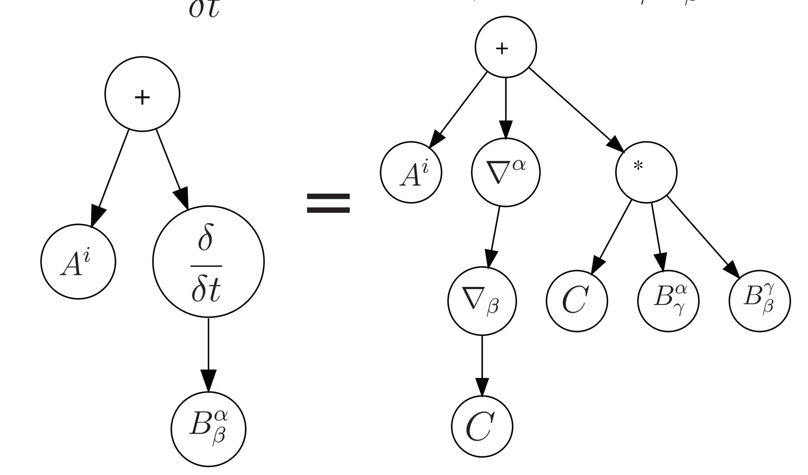
Method

We have built a prototype system to solve a subset of CMS problems. The System can be expanded to handle new rules and objects. Known solutions are compared to the output to determine if calculations are being handled correctly.

The overarching strategy of the system is to:
- Create a tree structure representing each expression.
- Walk the tree to find subtrees that match known rules in the CMS.
- Replace the subtrees with a new version representing the applied rule.
- Find equivalent subtrees to combine or cancel terms.
- Apply Rules towards a normal form.
- Export the final expression in symbolic or Mathematica form.

Applying Rules

$$A^i + \frac{\delta B^\alpha_\beta}{\delta t} = A^i + \nabla^\alpha\nabla_\beta C + C B^\alpha_\gamma B^\gamma_\beta$$



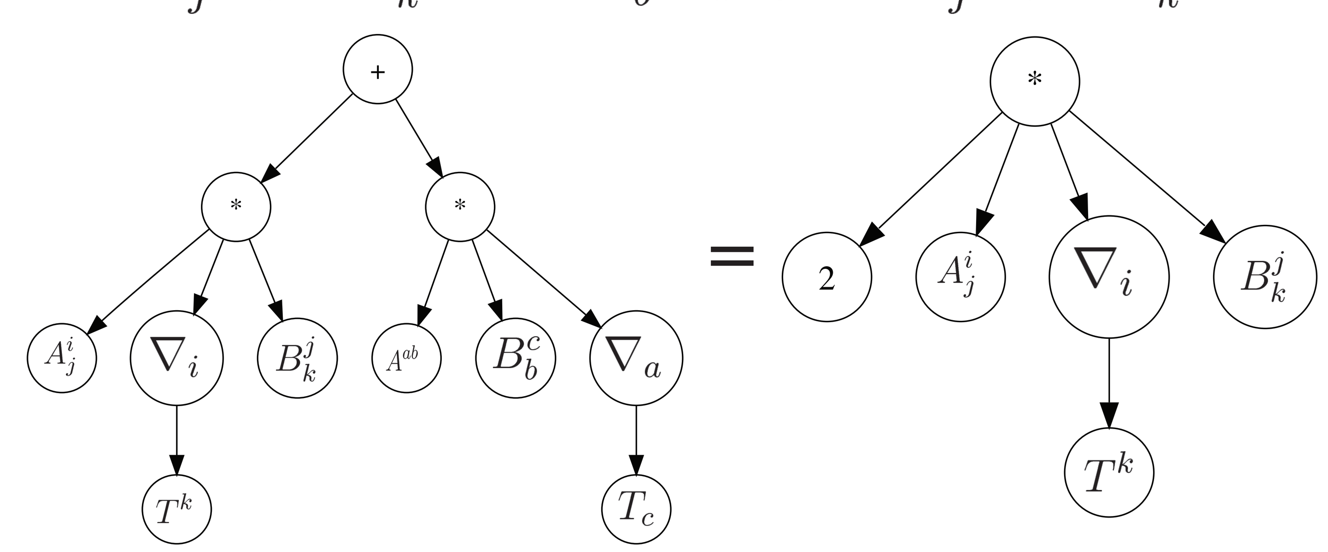
When a subtree that matches a rule is found, the existing subtree is pruned and a new subtree is created after applying the rule.
The figure shows the tree manipulation required to apply one rule for the invariant time derivative to the curvature tensor.

$$\frac{\delta B^\alpha_\beta}{\delta t} = \nabla^\alpha\nabla_\beta C + C B^\alpha_\gamma B^\gamma_\beta$$

These operations can all be applied automatically improving the speed and reliability of calculations. Some elements of the rule are variable, in this case the indexes alpha and beta can have any names as long as they have the correct properties and positions.
Many rules increase the total size of the expression. This creates long and complex expressions when working symbolically.

Combining Terms

$$A^i_j \nabla_i T^k B^j_k + A^{ab} B^c_b \nabla_a T_c = 2 A^i_j \nabla_i T^k B^j_k$$



The two products are equal, which means they can be combined to simplify the expression.
Why are these two products equal?

$$A^i_j \nabla_i T^k B^j_k = A^{ab} B^c_b \nabla_a T_c$$

Multiplication is commutative allowing term reordering.

$$A^i_j B^j_k \nabla_i T^k = A^{ab} B^c_b \nabla_a T_c$$

Contracted (paired) indexes can be renamed. i=a,j=b, k=c

$$A^a_b B^b_c \nabla_a T^c = A^{ab} B^c_b \nabla_a T_c$$

Contracted Indexes can be juggled. Flip the positions of the two b and c indexes. Possible positions are upper and lower.

$$A^{ab} B^c_b \nabla_a T_c = A^{ab} B^c_b \nabla_a T_c$$

Results

The system has been shown to successfully solve a subset of problems in the CMS. Problems were selected that have known solutions and the calculations of the system were compared to the results. The output was also converted into Mathematica code for numerical comparisons. Problems were selected that could be solved using the subset of rules currently implemented in the prototype system.

One model problem that was successfully solved by the system involved Poisson's equation. We examined Poisson's problem with $\nabla_i\nabla^i u = 1$ on a regular polygon with N sides. This problem uses all of the system's currently implemented rules.

The first variation of the model problem is determined by hand calculations. This gives the system a starting point to determine the higher variations.

$$u_1 = -CN_i\nabla^i u$$

The initial formula for higher order variations can be described by known formulas. Converting these into simplified form is performed automatically by the system.

$$u_2 = -CN_i\nabla^i u_1 - \frac{\delta C N^i \nabla_i u}{\delta t}$$

Allowing the program to automatically simply the expression results in the following expanded formula.

$$u_2 = -2CN^i\nabla_i u_1 - \frac{\delta C}{\delta t} N^i \nabla_i u + C Z^i_\alpha \nabla^\alpha C \nabla_i u - C^2 N^i N^j \nabla_i \nabla_j u$$

This expansion shows the number of terms is already increasing rapidly. The third variation can also be simplified in a similar manner.

$$u_3 = \left\{ \begin{array}{c} C^2\nabla^\beta C Z^i_\beta N^j \nabla_i\nabla_j u + 2C^2\nabla^\alpha C Z^i_\alpha N^j \nabla_i \nabla_j u \\ +C^2\nabla^\alpha C\nabla_i u Z^i_\beta B^\beta_\alpha - C^3 N^i N^j N^k \nabla_i\nabla_j\nabla_k u \\ +3C\nabla^\alpha C Z^i_\alpha \nabla_i u_1 + C\nabla^\beta \frac{\delta C}{\delta t} Z^i_\beta \nabla_i u \\ +2\frac{\delta C}{\delta t}\nabla^\alpha C Z^i_\alpha \nabla_i u - 3C^2 N^i N^j \nabla_i \nabla_j u_1 \\ +C\nabla_\beta C \nabla^\beta C N^i \nabla_i u - 3C\frac{\delta C}{\delta t} N^i N^j \nabla_i \nabla_j u \\ -3CN^i\nabla_i u_2 - \frac{\delta^2 C}{\delta t^2} N^i \nabla_i u - \frac{\delta C}{\delta t} N^i \nabla_i u_1 \end{array} \right\}$$

The third variation clearly makes the case for an automated approach to the symbolic computation. In this case, the third order has already become cumbersome to calculate by hand. Even higher orders, like the fifth and sixth can be calculated automatically by our prototype. Each of these results was converted to a general equation in polar coordinates to automatically compare the results to the known solutions. This model provides one of the many problems the system has already shown success on. The problem is also relevant to researchers in the field of CMS, giving a glimpse of the systems future potential.

Conclusions

The prototype system has been used to accurately solve a number of known problems. It has also been used to support ongoing research in the CMS. The system has been able to output the final results as Mathematica code allowing for numerical calculations.
The growth rate for many expressions has increased the number of terms to a size that overwhelms the system. Finding methods to better handle very large expressions will be a key factor moving forward.
Determining if expressions are equivalent is possible but currently takes factorial time. Finding more efficient methods for equivalence testing will improve system performance.
Overall, the system has shown significant promise and solved and number of significant model problems. Ongoing development will lead to a system that will simplify the research in the CMS.

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