A Symbolic Computation System for the Calculus of Moving Surfaces

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Abstract

The calculus of moving surfaces (CMS) provides analytic tools for finding solutions to a wide range of problems with moving surfaces including fluid film dynamics, boundary variation problems, and shape optimization problems. The CMS is an extension of tensor calculus on stationary surfaces to moving surfaces. As with any analytic framework, the complexity of calculations grows rapidly with the order of approximation. This quickly causes problems to become complex enough that hand calculations become error prone or intractable. A symbolic computation system will alleviate these problems, allowing researchers to examine problems that have not been previously solvable. No symbolic calculus system is currently available that supports the CMS. We have developed a prototype symbolic computation system that can solve boundary variation problems with the help of the CMS. Our system has been used to solve a series of model problems of interest to applied mathematicians.

The motivation is a boundary problem proposed by Grinfeld and Strang [2] in 2004. *What is the series in 1/N for the simple Laplace eigenvalues λ_N on a regular polygon with N sides?* In [2], the idea of expressing $\lambda_{N,n}$ as a series in 1/N was put forth and in [3] the first several terms were computed using the calculus of moving surfaces. The prototype implementation was successfully used to find an error in the fourth term in the series expansion in [3] which was previously computed by hand. The CMS approach to this problem is essentially the same as for Poisson's equation on polygons, which is used as the model problem in [1]. This problem involves Poisson's equation $\nabla_i \nabla^i u = 1$ and is therefore simpler than the eigenvalue equation $\nabla_i \nabla^i u = -\lambda u$ that is at the heart of the model problem. The fundamental simplification is unrelated to the CMS: it comes from the fact that all solution variations u_n (where n is the order of the variation) satisfy Laplace's equation in the interior. This makes it easy to solve for u_n and use the result in the next order of variation.

The expression for E_1 and the equation for u_1 are obtained analytically. Higher order variations follow by direct application of the rules of CMS. The second order variation u_2 is governed by the boundary value system

$$\nabla_i \nabla^i u_2 = 0 \tag{1}$$

$$u_2|_S = -2CN^i \nabla_i u_1 - \frac{\delta C}{\delta t} N^i \nabla_i u + CZ^i_\alpha \nabla^\alpha C \nabla_i u - C^2 N^i N^j \nabla_i \nabla_j u \tag{2}$$

The second order energy variation E_2 is given by

$$E_2 = -\frac{1}{2}\int_S \left(-\frac{\delta C}{\delta \tau} \nabla_i u \nabla^i u - 2C \nabla_i u_1 \nabla^i u - 2C^2 N^i \nabla_i \nabla_j u \nabla^j u + C^2 B^\alpha_\alpha \nabla_i u \nabla^i u \right) dS \tag{3}$$

and third energy variation E_3 is given by

$$E_3 = \frac{1}{2}\left\{ \begin{array}{c} -C^3 B^\alpha_\alpha B^\beta_\beta \nabla_i u \nabla^i u + 3C\frac{\delta C}{\delta t} B^\beta_\beta \nabla_i u \nabla^i u \\ +2C^2 B^\alpha_\alpha \nabla^i u_1 \nabla_i u + 2C^3 B^\alpha_\alpha N^l \nabla^i u \nabla_l \nabla_i u \\ -\frac{\delta^2 C}{\delta^2 t} \nabla_i u \nabla^i u - 2\frac{\delta C}{\delta t} \nabla_i u_1 \nabla^i u \\ -2C \nabla_i u_2 \nabla^i u - C^2 N^j \nabla^i u \nabla_j \nabla_i u_1 \\ -2C \nabla^i u_1 \nabla_i u_1 - C^2 N^j \nabla_i u_1 \nabla_j \nabla^i u \\ -C^2 N^j \nabla_i u \nabla_j \nabla^i u_1 \\ -2\frac{\delta C}{\delta t} \nabla_i u_1 \nabla^i u - C^2 N^j \nabla^i u_1 \nabla_j \nabla_i u \\ +C^2 \nabla^\alpha \nabla_\alpha C \nabla_i u \nabla^i u + C^3 B^\alpha_\beta B^\beta_\alpha \nabla_i u \nabla^i u \\ +2C^2 B^\alpha_\alpha \nabla_i u_1 \nabla^i u + 2C^3 B^\alpha_\alpha N^k \nabla^i u \nabla_k \nabla_i u \\ -3C\frac{\delta C}{\delta t} N^i \nabla^j u \nabla_i \nabla_j u + C^2 \nabla^\alpha C Z^j_\alpha \nabla^i u \nabla_i \nabla_j u \\ -C^2 N^j \nabla_j \nabla_i u \nabla^i u_1 - C^3 N^i N^j \nabla_i \nabla_k u \nabla_j \nabla^k u \\ -C^2 N^k \nabla_i \nabla_j u_1 \nabla^i u - 2C^3 N^j N^k \nabla_k \nabla_i \nabla_j u \nabla^i u \\ -2C\frac{\delta C}{\delta t} N^j \nabla_i u \nabla_j \nabla^i u + C^2 \nabla_i u \nabla_j \nabla^i u Z^j_\alpha \nabla^\alpha C \\ -C^2 N^j \nabla_i u_1 \nabla_j \nabla^i u - C^3 N^j N^k \nabla_j \nabla^i u \nabla_k \nabla_i u \\ -\frac{\delta C}{\delta t} C N^m \nabla_m \nabla_i u \nabla^i u - C^2 N^j \nabla_j \nabla^i u_1 \nabla_i u \end{array} \right\}. \tag{4}$$

Equation (4), like no other, makes the case for the symbolic calculus of moving surfaces. While each element can be evaluated in straightforward fashion, the sheer number of these elements is overwhelming.

Our prototype system has been able to calculate solutions up to the fifth order variation automatically. These problems have already shown that the system can evaluate high order boundary variations for complex surface motions.

References

[1] M. Boady, P. Grinfeld, and J.Johnson. Boundary variation of Poisson's equation: a model problem for symbolic calculus of moving surfaces. *Submitted to Numer. Funct. Anal. Opt.*, 2010

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[3] P. Grinfeld and G. Strang. Laplace eeigenvalues on regular polygons: a series in 1/N. *Submitted to Transactions AMS*, 2010.