CS XXX - Quantum Worksheet

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1 Introduction

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Complete this worksheet in any way you like.
This assignment is worth 100

2 Quantum Worksheet

Question 1 : 5 points

In classical logic, we use Boolean Values 0 and 1. Classical logic circuits can only take these two values.

In a Quantum Circuit, a **qubit** (Quantum Bit) can have many values. To represent the **state** of a **qubit** we will need complex numbers.

This lab provides an overview of the important properties of Complex Numbers. This lab only presents introductory material. As we progress into Quantum Circuits, we will introduce more properties as needed.

We will start with the most straightforward appearance of Complex Numbers.

A quadratic expression has the form

$$ax^2 + bx + c = 0 \tag{1}$$

The solution to this equation is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2}$$

(a) (5 points) Confirm this formula is correct by plugging Equation 2 into Equation 1.
All terms should cancel, giving a result $0 = 0$.

Question 2 : 6 points

There are three different types of outcomes for the following formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

- $(b^2 - 4ac > 0) \implies$ Two Real Solutions
- $(b^2 - 4ac = 0) \implies$ One Real Solution
- $(b^2 - 4ac < 0) \implies$ No Real Solutions

When the discriminant is negative, there are no real solutions because there is no real answer to $\sqrt{-1}$. This is were complex numbers are introduced.

Let,

$$i = \sqrt{-1} \tag{4}$$

Since $i^2 = -1$ we can now find additional solutions to the Quadratic Equation. Numbers involving i are called Complex Numbers.

For example,

$$2x^2 + 3x + 2 = 0 \tag{5}$$

We have

$$a = 2 \tag{6}$$
$$b = 3 \tag{7}$$
$$c = 2 \tag{8}$$
$$x = \frac{-3 \pm \sqrt{3^2 - 4 * 2 * 2}}{2 * 2} \tag{9}$$
$$= \frac{-3 \pm \sqrt{9 - 16}}{4} \tag{10}$$
$$= \pm \frac{\sqrt{9 - 16}}{4} - \frac{3}{4} \tag{11}$$
$$= \pm \frac{\sqrt{-7}}{4} - \frac{3}{4} \tag{12}$$
$$= \pm \frac{i\sqrt{7}}{4} - \frac{3}{4} \tag{13}$$

This equation has two Complex solutions $\frac{i\sqrt{7}}{4} - \frac{3}{4}$ and $-\frac{i\sqrt{7}}{4} - \frac{3}{4}$.

Give all solutions to each of the following equations.

(a) (1 point) $6x^2 + 4x + 5 = 0$

(b) (1 point) $6x^2 + x + 6 = 0$

(c) (1 point) $-6x^2 - x - 2 = 0$

(d) (1 point) $2x^2 - x - 5 = 0$

(e) (1 point) $-2x^2 - 6x - 4 = 0$

(f) (1 point) $x^2 + 2x + 1 = 0$

Question 3 : 9 points

By definition we know $i^0 = 1$, $i^1 = i$, and $i^2 = -1$.

Evaluate each of the following.

(a) (1 point) i^3

(b) (1 point) i^4

(c) (1 point) i^5

(d) (1 point) i^6

(e) (1 point) i^7

(f) (1 point) i^8

(g) (1 point) i^9

(h) (2 points) There are only 4 possible outcomes. Give Psuedocode for a function `imagineExp(a)` that computes i^a for $a \geq 0$.

Question 4 : 6 points

There is an important relationship between imaginary numbers and **Euler's Number** e. The number e is irrational, meaning it cannot be represented by a decimal number with finite digits.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \cdots \approx 2.71828 \tag{14}$$

This constant is very important in mathematics, but the following equation will be of most importance to Quantum Circuits.

$$e^{ix} = \cos x + i \sin x \tag{15}$$

We can convert e^{ix} expressions into a more standard complex number format.

$$e^{9i} = -0.9111 + 0.4121i \tag{16}$$

Approximate the following to 4 decimal digits.

(a) (1 point) e^{12i}

(b) (1 point) e^{-4i}

(c) (1 point) $e^{\pi i}$

(d) (1 point) e^{ei}

(e) (1 point) $e^{\frac{i}{2}}$

(f) (1 point) e^{i}

Question 5 : 6 points

We will look at some additional basic operations on complex numbers.

Description	Expression	Evaluation
Addition	$(x+yi)+(a+bi)$	$(x+a)+(y+b)i$
Subtraction	$(x+yi)-(a+bi)$	$(x-a)+(y-b)i$
Multiplication	$(x+yi)(a+bi)$	$(ax-yb)+(ay+xb)i$
Absolute Value	$\lvert(x+yi)\rvert$	$\sqrt{x^2+y^2}$
Reciprocal	$\frac{1}{x+yi}$	$\frac{x}{x^2+y^2}+\frac{-y}{x^2+y^2}i$
Complex Conjugate	$(x+yi)^*$	$x-yi$

(a) (3 points) Show that for any complex number $z = x + yi$ we have $(z)(z^*) = |z|^2$. Show your work.

(b) (3 points) Simplify the expression $\frac{x+yi}{a+bi}$ to find an expression for division. Show your work.

Question 6 : 2 points

We can represent complex numbers geometrically. Each value $x + yi$ can be represented as a point on a circle.

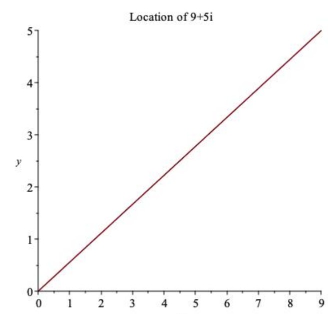
$$r = |z| \tag{17}$$
$$\theta = \tan^{-1}(y/x) \tag{18}$$

If we take the point $9 + 5i$, we can plot it as a point. It is a point at $x = 9$ and $y = 5$.

We can also reach the same point by starting at $(0, 0)$ and moving to a point (r, θ).

We move θ radians and out radius r.

$$r = |9 + 5i| = \sqrt{9^2 + 5^2} = 56.62155 \tag{19}$$
$$\theta = \tan^{-1} \frac{5}{9} = 0.50798 \text{ Radians} \tag{20}$$



(a) (1 point) Compute r and θ for $2 - 3i$ (approximate to 4 digits).

(b) (1 point) Compute r and θ for e^{4i} (appromixate to 4 digits).

Question 7 : 3 points

In classical logic, a circuit is made from boolean logic operators. In a quantum computer, operators (or gates) are matrices. The state of a quantum circuit is a vector and each gate is a matrix.

In this lab, we will review the concepts from linear algebra required to model a quantum circuit.

A qubit (quantum bit) still has values 0 and 1 like a classical computer. Unlikely a classical computer, it can be in a super-position state. In this state, the qubit it both 0 and 1. When the value of the qubit is checked, we only read a 0 or 1. The probability of reading each value if determined by the super-position. In the next lab, we will look at the mechanics of a qubit and how this works.

To represent a qubit, we will use a vector. The first (upper) position will represent 0 and the lower will represent 1. In the below vector, we have a 0 bit.

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{21}$$

This vector is in a superposition between 0 and 1. It will have a 50% chance of being 0 and a 50% chance of being 1.

$$v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{22}$$

For this lab, we will just look at general vectors.

$$v = \begin{bmatrix} a \\ b \end{bmatrix} \tag{23}$$

The length of a vector is given by the formula $|v| = \sqrt{a^2 + b^2}$.

Compute the length of the following vectors.

(a) (1 point) $\left| \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right|$

(b) (1 point) $\left| \begin{bmatrix} 4i \\ 2 \end{bmatrix} \right|$

(c) (1 point) $\left| \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right|$

Question 8 : 4 points

A matrix is a transform on a vector. It takes a vector as input and returns a vector as output. We use be working with square matrix, meaning the input vector will always have the same number of elements as the output vector.

Below is a general 2 by 2 matrix.

$$M = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \tag{24}$$

Matrix-Vector Multiplication follows the following rule.

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} aw + bx \\ ay + bz \end{bmatrix} \tag{25}$$

(a) (1 point) Compute $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

(b) (1 point) Compute $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

(c) (1 point) Compute $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

(d) (1 point) Compute $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Question 9 : 5 points

The transpose M^T of a matrix is formed turning all rows into columns and columns into rows.

$$M = \begin{bmatrix} w & x \\ y & z \end{bmatrix}, M^T = \begin{bmatrix} w & y \\ x & z \end{bmatrix} \tag{26}$$

(a) (1 point) What is the transpose of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) (1 point) What is the transpose of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(c) (1 point) What is the transpose of $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

(d) (1 point) What is the transpose of $\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

(e) (1 point) The Hermitian Conjugate $M^\dagger$ is the transpose with the complex conjugate of every number.

What is the Hermitian Conjugate of $\begin{bmatrix} 2+4i & 3+i \\ 7-2i & 9i \end{bmatrix}$

Question 10 : 8 points

There are two ways to multiply matrix. First, we can imagine each column as a vector. Then use matrix-vector multiplication to come up with the new row.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix} \tag{27}$$

(a) (2 points) Compute $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix}$

(b) (2 points) Compute $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$

(c) (2 points) Compute $\begin{bmatrix} i & -i \\ -i & i \end{bmatrix} \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$

(d) (2 points) Compute $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Question 11 : 8 points

Here are 5 important Matrices.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{28}$$

(a) (2 points) Show that $HH^\dagger = I$.

(b) (2 points) Show that $XX^\dagger = I$.

(c) (2 points) Show that $YY^\dagger = I$.

(d) (2 points) Show that $ZZ^\dagger = I$.

Question 12 : 9 points

The **Tensor Product**, $\otimes$, is a second kind of matrix multiplication. This time, we create a larger matrix.

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \otimes \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix} \tag{29}$$

(a) (3 points) Compute $H \otimes I$.

(b) (3 points) Compute $X \otimes Y$.

(c) (3 points) Compute $X \otimes Z$

Question 13 : 4 points

A bit is the fundamental unit of data on a classical computer. A bit can be either of two values, it can be 0 or 1. A bit on a classical computer technically has three states. The bit is a electrical current passed through the computer on wires. A specific range of voltages is read as 0. Another range is read as 1. There is a slight gap in the middle where the result is unclear and therefore useless. When a bit changes state, it briefly passes by this middle ground, but this information is useless.

In a Quantum Computer, a bit is a quantum object like and atom. An atom can be in a ground state (we will call 0) or in an excited state (we will call 1). There are numerous Quantum particles that can be used in Quantum Computers. They all the the same basic abstract concepts.

There is one very important difference between Quantum and classical physics when it comes to circuits. A Quantum particle can be in multiple state concurrently, this is called superposition. When we **read** the value of the quantum particle, we will only see a 1 or a 0. While the Quantum particle is traveling through the circuit it can meaningfully be in a superposition state.

We call a quantum bit a **qubit**. A qubit is represented by a **linear combination** of complex numbers.

$$\alpha |0\rangle + \beta |1\rangle \tag{30}$$

When we measure the qubit, we read 0 with a probability $|\alpha|^2$. We will read 1 with a probability $|\beta|^2$. Since there is a 100% chance of reading either 0 or 1, we have $|\alpha|^2 + |\beta|^2 = 1$.

A simple example qubit is

$$1 |0\rangle + 0 |1\rangle \tag{31}$$
$$|\alpha|^2 = 1^2 = 1 \tag{32}$$
$$|\beta|^2 = 0^2 = 0 \tag{33}$$

This qubit has a 100% chance of being read as a 1 and a 0% chance of being 0. It will act just like a classical bit set to 1.

Determine the probabilities of each of the following qubits.

(a) (1 point) $0 |0\rangle + 1 |1\rangle$

(b) (1 point) $\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$

(c) (1 point) $\frac{1}{\sqrt{3}} |0\rangle + \frac{\sqrt{2}}{\sqrt{3}} |1\rangle$

(d) (1 point) $i |0\rangle + 0 |1\rangle$

Question 14 : 8 points

In a classical multi-bit system, each bit can be 0 or 1. This leads to the classic truth table for the system. Below is a Truth Table for a simple 2-bit circuit (an XOR).

A	B	Result
0	0	0
0	1	1
1	0	1
1	1	0

In a classical computer, we would state that if the system starts with $A = 0$ and $B = 1$, then the result is 1. In a quantum computer, this is not the case. Every quantum gate must be reversible. Although we may not read every bit, all bits must be represented at both the start and end of the circuit. We would represent the same Truth Table circuit by saying the input is 010 (the last bit is for the result) and ends with the output 011. We may only read the third bit, but all three bits must exist as both inputs and outputs.

A 2-qubit system would be represented as a linear combination of 4 values.

$$\alpha \left|00\right\rangle + \beta \left|01\right\rangle + \gamma \left|10\right\rangle + \kappa \left|11\right\rangle \tag{34}$$

(a) (2 points) Come up with values for α, β, γ, and κ that make each outcome have the same probability (25% each)

(b) (2 points) Come up with values for α, β, γ, and κ such that there is a 100% chance the outcome is 00.

(c) (2 points) Come up with values for α, β, γ, and κ such that there is a 50% chance the outcome is 00 and 50% chance of 10.

(d) (2 points) Come up with values for α, β, γ, and κ such that there is a 90% chance of 10 and 10% chance of 01.

Question 15 : 5 points

Since each possible outcome is represented by a complex number, the Quantum System is storing a larger amount of information then a classical computer. In a classical computer, a 4-bit system has 4 literal bits.

We will briefly look at how much data is required to simulate a qubit system on a classical computer. This will give us an idea of the amount of extra information stored in the system.

A classical computer cannot store a *true* complex number. For example, $\frac{i}{\sqrt{\pi}}$ would require an infinite number of bits. Let's assume a complex number is a pair of approximated floating point numbers. Let us also assume the each floating point value is represented to 64-bits.

A 2-qubit system has 4 possible values ($2^2 = 4$ values 00, 01, 10, 11). Each of these values needs a complex number ($2 * 64 = 128$ bits). This means we would need at minimum $4 * 128 = 512$ bits to just store the expression for the systems result.

In general, for a n qubit system, we can simulate it with $2^n\,(128)$ bits.

Figure out how many classical bits it would take to store expression for the quantum system.

Give the result in the most meaningful classical measurement.

byte (B)	8 bits
Kilobytes (kB)	1000 B
Megabytes (MB)	1000 kB
Gigabytes (GB)	1000 MB
Terabytes (TB)	1000 GB
Petabytes (PB)	1000 TB
Exabytes (EB)	1000 PB

(a) (1 point) How much memory would be needed to represent a 8-qubit system?

(b) (1 point) How much memory would be needed to represent a 16-qubit system?

(c) (1 point) How much memory would be needed to represent a 32-qubit system?

(d) (1 point) How much memory would be needed to represent a 46-qubit system?

(e) (1 point) It is currently estimated that there are 10^{82} atoms in the observable universe. Imagine each of these atoms was used to store a single classical bit of computer memory.
How many qubits could this system represent? (approximate $10^{82} = 2^n\,(128)$)

Question 16 : 4 points

When working with qubits, it is easier to represent the linear combination as a vector. Just be careful to keep track of what each number represents.

$$\alpha|0\rangle + \beta|1\rangle = \begin{bmatrix}\alpha \\ \beta\end{bmatrix} \tag{35}$$

Quantum Gates are Matrix Operations on vectors.

A simple 1-qubit gate is the Pauli-X Gate.

$$X = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix} \tag{36}$$

We start with a qubit in the state $1|0\rangle + 0|1\rangle$.

$$\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} = \begin{bmatrix}0 \\ 1\end{bmatrix} \tag{37}$$

The bit has been flipped from 0 to 1. This is the same as the classic NOT gate.

(a) (2 points) Show that $X(0|0\rangle + 1|1\rangle) = 1|0\rangle + 0|1\rangle$.

(b) (2 points) In classical logic circuits, a double negative cancels out. $\neg\neg A = A$
Show that this is also true for the X gate. $X(X(\alpha|0\rangle + \beta|1\rangle)) = \alpha|0\rangle + \beta|1\rangle$

Question 17 : 8 points

The Hadamard Gate is another of the most important gates in Quantum Computing.

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{38}$$

(a) (2 points) What is the probabilities of the system $H(1\,|0\rangle + 0\,|1\rangle)$?

(b) (2 points) What is the probabilities of the system $H(0\,|0\rangle + 1\,|1\rangle)$?

(c) (1 point) There is a small difference between the results of (a) and (b). What is it?

(d) (3 points) What happens when the Hadamard gate is applied to a qubit twice?
$H(H(\alpha\,|0\rangle + \beta\,|1\rangle)) =?$

3 Further Reading

https://www2.clarku.edu/faculty/djoyce/complex/

http://mathworld.wolfram.com/ComplexNumber.html

https://en.wikipedia.org/wiki/E_(mathematical_constant)

https://www.math10.com/en/geometry/vectors-operations/vectors-operations.html

https://www.varsitytutors.com/hotmath/hotmath_help/topics/multiplying-vector-by-a-matrix

https://en.wikipedia.org/wiki/Matrix_multiplication

https://www.mathwords.com/t/transpose_of_a_matrix.htm

https://en.wikipedia.org/wiki/Tensor_product